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# Limited operators and differentiability.

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**Abstract.** We characterize the limited operators by differentiability of convex continuous functions. Given Banach spaces  $Y$  and  $X$  and a linear continuous operator  $T : Y \longrightarrow X$ , we prove that  $T$  is a limited operator if and only if, for every convex continuous function  $f : X \longrightarrow \mathbb{R}$  and every point  $y \in Y$ ,  $f \circ T$  is Fréchet differentiable at  $y \in Y$  whenever  $f$  is Gâteaux differentiable at  $T(y) \in X$ .

**Keyword, phrase:** Limited operators, Gâteaux differentiability, Fréchet differentiability, convex functions, extreme points.

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## 1 Introduction.

A subset  $A$  of a Banach space  $X$  is called limited, if every weak\* null sequence  $(p_n)_n$  in  $X^*$  converges uniformly on  $A$ , that is,

$$\lim_{n \rightarrow +\infty} \sup_{x \in A} |\langle p_n, x \rangle| = 0.$$

We know that every relatively compact subset of  $X$  is limited, but the converse is false in general. A bounded linear operator  $T : Y \longrightarrow X$  between Banach spaces  $Y$  and  $X$  is called limited, if  $T$  takes the closed unit ball  $B_Y$  of  $Y$  to a limited subset of  $X$ . It is easy to see that  $T : Y \longrightarrow X$  is limited if and only if, the adjoint operator  $T^* : X^* \longrightarrow Y^*$  takes weak\* null sequence to norm null sequence. For useful properties of limited sets and limited operators we refer to [11], [4], [6] and [1].

We know that in a finite dimensional Banach space, the notions of Gâteaux and Fréchet differentiability coincide for convex continuous functions. In [5], Borwein and Fabian proved that a Banach space  $Y$  is infinite dimensional if and only if, there exists on  $Y$  functions  $f$  having points at which  $f$  is Gâteaux but not Fréchet differentiable. They also pointed in the introduction of [5] the observation that if the sup-norm  $\|\cdot\|_\infty$  on  $c_0$  is Gâteaux differentiable at some point, then it is Fréchet differentiable there. In this article we observe that this phenomenon is not just related to the sup-norm but more generally, for each convex lower semicontinuous function  $g : l^\infty \rightarrow \mathbb{R} \cup \{+\infty\}$ , if  $g$  is Gâteaux differentiable at some point  $a \in c_0$  which is in the interior of its domain, then the restriction of  $g$  to  $c_0$  is Fréchet differentiable at  $a$ . This hold in particular when  $g = (f^*)^*$  is the Fenchel biconjugate of a convex continuous function  $f : c_0 \rightarrow \mathbb{R}$ . In fact, this phenomenon is due, (see Corollary 1 in the Appendix and the comment just before), to the fact that the canonical embedding  $i : c_0 \longrightarrow l^\infty$  is a limited operator (see the reference [6]).

The goal of this paper, is to prove the following characterization of limited operators in terms of the coincidence of Gâteaux and Fréchet differentiability of convex continuous functions.

**Theorem 1.** *Let  $Y$  and  $X$  be two Banach spaces and  $T : Y \longrightarrow X$  be a continuous linear operator. Then,  $T$  is a limited operator if and only if, for every convex continuous function  $f : X \longrightarrow \mathbb{R}$  and every  $y \in Y$ , the function  $f \circ T$  is Fréchet differentiable at  $y \in Y$  whenever  $f$  is Gâteaux differentiable at  $T(y) \in X$ .*

As consequence we give, in Theorem 2 below, new characterizations of infinite dimensional Banach spaces, complementing a result of Borwein and Fabian in [[5], Theorem 1.].

A real valued function  $f$  on a Banach space will be called a PGNF-function (see [5]) if there exists a point at which  $f$  is Gâteaux but not Fréchet differentiable. A JN-sequence (due to Josefson-Nissenzweig theorem, see [[7], Chapter XII]) is a sequence  $(p_n)_n$  in a dual space  $Y^*$  that is weak\* null and  $\inf_n \|p_n\| > 0$ . We say that a function  $g$  on  $X^*$  has a norm-strong minimum (resp. weak\*-strong minimum) at  $p \in X^*$  if  $g(p) = \inf_{q \in X^*} g(q)$  and  $(p_n)_n$  norm converges (resp. weak\* converges) to  $p$  whenever  $g(p_n) \longrightarrow g(p)$ . A norm-strong minimum and weak\*-strong minimum are in particular unique.

**Theorem 2.** *Let  $Y$  be a Banach space. Then the following assertions are equivalent.*

- (1)  $Y$  is infinite dimensional.
- (2) There exists a JN-sequence in  $Y^*$ .
- (3) There exists a convex norm separable and weak\* compact metrizable subset  $K$  of  $Y^*$  containing 0 and a continuous seminorm  $h$  on  $X^*$  which is weak\* lower semicontinuous and weak\* sequentially continuous, such that the restriction  $h|_K$  has a weak\*-strong minimum but not norm-strong minimum at 0.
- (4) There exists a Banach space  $X$  and a linear continuous non-limited operator  $T : Y \longrightarrow X$ .
- (5) There exists on  $Y$  a convex continuous PGNF-function.

In Section 2 we give some preliminary results, specially the key Lemma 2. In Section 3, we give the proof of Theorem 1 (divided into two part, Theorem 3 and Theorem 4) and the proof of Theorem 2. In Section 4 we give some complementary remarks.

## 2 Preliminaries.

We recall the following classical result.

**Lemma 1.** *Let  $\mathcal{T}_1$  and  $\mathcal{T}_2$  be topologies on a set  $K$  such that*

- (1)  $K$  is Hausdorff with respect  $\mathcal{T}_1$ ,
- (2)  $K$  is compact with respect to  $\mathcal{T}_2$ ,
- (3)  $\mathcal{T}_1 \subset \mathcal{T}_2$ .

*Then  $\mathcal{T}_1 = \mathcal{T}_2$ .*

*Proof.* Let  $F \subset K$  be a  $\mathcal{T}_2$ -closed set. It follows that  $F$  is  $\mathcal{T}_2$ -compact, since  $K$  is  $\mathcal{T}_2$ -compact. Let  $\{\mathcal{O}_i : i \in I\}$  be any cover of  $F$  by  $\mathcal{T}_1$ -open sets. Since  $\mathcal{T}_1 \subset \mathcal{T}_2$ , then each of these sets is also  $\mathcal{T}_2$ -open. Hence, there exist a finite subcollection that covers  $F$ . It follows that  $F$  is  $\mathcal{T}_1$ -compact and therefore is  $\mathcal{T}_1$ -closed since  $\mathcal{T}_1$  is Hausdorff. This implies that  $\mathcal{T}_2 \subset \mathcal{T}_1$ . Consequently,  $\mathcal{T}_1 = \mathcal{T}_2$ . □

Now, we establish the following useful lemma. If  $B$  is a subset of a dual Banach space  $X^*$ , we denote by  $\overline{\mathcal{O}}^{w^*}(B)$  the weak\* closed convex hull of  $B$ .

**Lemma 2.** *Let  $X$  be a Banach space and  $K$  be a subset of  $X^*$ .*

(1) *Suppose that  $K$  is norm separable, then there exists a sequence  $(x_n)_n$  in the unit sphere  $S_X$  of  $X$  which separate the points of  $K$  i.e. for all  $p, p' \in K$ , if  $\langle p, x_n \rangle = \langle p', x_n \rangle$  for all  $n \in \mathbb{N}$ , then  $p = p'$ . Consequently, if  $K$  is a weak\* compact and norm separable set of  $X^*$ , then the weak\* topology of  $X^*$  restricted to  $K$  is metrizable.*

(2) *Let  $(p_n)_n$  be a weak\* null sequence in  $X^*$ . Then, the set  $\overline{co}^{w*} \{p_n : n \in \mathbb{N}\}$  is convex weak\* compact and norm separable.*

*Proof.* (1) Since  $K$  is norm separable, then  $K - K := \{a - b / (a, b) \in K \times K\}$  is also norm separable and so there exists a sequence  $(q_n)_n$  of  $K - K$  which is dense in  $K - K$ . According to the Bishop-Phelps theorem [3], the set

$$D = \{r \in X^* \mid r \text{ attains its supremum on the sphere } S_X\}$$

is norm-dense in the dual  $X^*$ . Thus, for each  $n \in \mathbb{N}$ , there exists  $r_n \in D$  such that  $\|q_n - r_n\| < \frac{1}{1+n}$ . For each  $n \in \mathbb{N}$ , let  $x_n \in S_X$  be such that  $\|r_n\| = \langle r_n, x_n \rangle$ . We claim that the sequence  $(x_n)_n$  separate the points of  $K$ . Indeed, let  $q \in K - K$  and suppose that  $\langle q, x_n \rangle = 0$ , for all  $n \in \mathbb{N}$ . There exists a subsequence  $(q_{n_k})_k \subset K - K$  such that  $\|q_{n_k} - q\| < \frac{1}{k}$  for all  $k \in \mathbb{N}^*$  and so we have  $\|r_{n_k} - q\| < \frac{1}{1+n_k} + \frac{1}{k}$ . It follows that

$$\begin{aligned} \|r_{n_k}\| &= \langle r_{n_k}, x_{n_k} \rangle \\ &= \langle r_{n_k}, x_{n_k} \rangle - \langle q, x_{n_k} \rangle \\ &\leq \|r_{n_k} - q\| \\ &< \frac{1}{1+n_k} + \frac{1}{k}. \end{aligned}$$

Hence, for all  $k \in \mathbb{N}^*$ ,  $\|q\| \leq \|q - r_{n_k}\| + \|r_{n_k}\| < 2(\frac{1}{1+n_k} + \frac{1}{k})$ , which implies that  $q = 0$ , and so that  $(x_n)_n$  separate the points of  $K$ . Now, suppose that  $K$  is weak\* compact subset of  $X^*$ . We show that the weak\* topology of  $X^*$  restricted to  $K$  is metrizable. Indeed, each  $x \in X$  determines a seminorm  $\nu_x$  on  $X^*$  given by

$$\nu_x(p) = |\langle p, x \rangle|, \quad p \in X^*.$$

The family of seminorms  $(\nu_x)_{x \in X}$  induces the weak\* topology  $\sigma(X^*, X)$  on  $X^*$ . The subfamily  $(\nu_{x_n})_n$  also induces a topology on  $X^*$ , which we will call  $\mathcal{T}$ . Since this is a smaller family of seminorms, we have  $\mathcal{T} \subseteq \sigma(X^*, X)$ . Suppose that  $p, p' \in K$  and  $\nu_{x_n}(p - p') = 0$  for all  $n \in \mathbb{N}$ . Then we have  $\langle p, x_n \rangle = \langle p', x_n \rangle$  for all  $n \in \mathbb{N}$  and so we have that  $p = p'$  since  $(x_n)_n$  separates the points of  $K$ . Consequently,  $K$  is Hausdorff with respect to the topology  $\mathcal{T}|_K$  (the restriction of  $\mathcal{T}$  to  $K$ ). Thus  $\mathcal{T}|_K$  is a Hausdorff topology on  $K$  induced from a countable family of seminorms, so this topology is metrizable. More precisely,  $\mathcal{T}|_K$  is induced from the metric

$$d(p, p') := \sum_{n=0}^{+\infty} 2^{-n} \frac{\nu_{x_n}(p - p')}{1 + \nu_{x_n}(p - p')}.$$

Then we have that  $K$  is Hausdorff with respect to  $\mathcal{T}|_K$ , and is compact with respect to  $\sigma(X^*, X)|_K$ . Lemma 1 implies that  $\mathcal{T}|_K = \sigma(X^*, X)|_K$ . Hence  $\sigma(X^*, X)|_K$  is metrizable.

(2) Let  $(p_n)_n$  be a weak\* null sequence in  $X^*$  and set  $K = \overline{co}^{w*} \{p_n : n \in \mathbb{N}\}$ . Clearly  $K$  is a convex and weak\* compact subset of  $X^*$ . According to Haydon's theorem [[8], Theorem 3.3] the weak\* compact convex set  $K$  is the norm closed convex hull of its extreme points whenever  $ex(K)$  (the set of extreme points of  $K$ ) is norm separable. By the Milman theorem [[10], p.9]  $ex(K) \subset \overline{\{p_n : n \in \mathbb{N}\}}^{w*} = \{p_n : n \in \mathbb{N}\} \cup \{0\}$  so that  $ex(K)$  is norm separable and, hence, by Haydon's theorem,  $K$  itself is weak\* compact, convex, and norm separable.  $\square$

The following proposition will be used in the proof of Theorem 3.

**Proposition 1.** *Let  $X$  be a Banach space and  $K$  be a weak\* compact and norm separable subset of  $X^*$  containing 0. Then, there exists a continuous seminorm  $h$  on  $X^*$  satisfying*

- (1)  *$h$  is weak\* lower semicontinuous and sequentially weak\* continuous,*
- (2) *the restriction  $h|_K$  of  $h$  to  $K$  has a weak\*-strong minimum at 0.*

*Proof.* Using Lemma 2, there exists a sequence  $(x_k)_k \subset S_X$  which separate the points of  $K$ . Define the function  $h : X^* \rightarrow \mathbb{R}$  as follows:

$$h(x^*) = \left( \sum_{k \geq 0} 2^{-k} (\langle x^*, x_k \rangle)^2 \right)^{\frac{1}{2}}, \quad \forall x^* \in X^*.$$

It is clear that  $h$  is a seminorm, and since  $h(x^*) \leq \|x^*\|$  for all  $x^* \in X^*$ , it is also continuous. Since  $h$  is the supremum of a sequence of weak\* continuous functions, it is weak\* lower semicontinuous. On the other hand, since the series  $\sum_{k \geq 0} 2^{-k} (\langle x^*, x_k \rangle)^2$  uniformly converges on bounded sets of  $X^*$  and since the maps  $\hat{x}_k : x^* \mapsto \langle x^*, x_k \rangle$  are weak\* continuous for all  $k \in \mathbb{N}$ , then  $h$  is sequentially weak\* continuous. If  $p \in K$  and  $h(p) = 0$ , then  $\langle p, x_k \rangle = 0$  for all  $k \in \mathbb{N}$  which implies that  $p = 0$ , since the sequence  $(x_k)_k$  separate the points of  $K$ . Hence, the restriction of  $h$  to  $K$  has a unique minimum at 0. This minimum is necessarily a weak\*-strong minimum since  $K$  is weak\* metrizable by Lemma 2, this follows from a general fact which says that for every lower semicontinuous function on a compact metric space  $(K, d)$ , a unique minimum is necessarily a strong minimum for the metric  $d$  in question.  $\square$

### 3 Limited operators and differentiability.

Recall that the domain of a function  $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ , is the set

$$\text{dom}(f) := \{x \in X / f(x) < +\infty\}.$$

For a function  $f$  with  $\text{dom}(f) \neq \emptyset$ , the Fenchel transform of  $f$  is defined on the dual space for all  $p \in X^*$  by

$$f^*(p) := \sup_{x \in X} \{\langle p, x \rangle - f(x)\}.$$

The second transform  $(f^*)^*$  is defined on the bidual  $X^{**}$  by the same formula. We denote by  $f^{**}$ , the restriction of  $(f^*)^*$  to  $X$ , where  $X$  is identified to a subspace of  $X^{**}$  by the canonical embedding. Recall that the Fenchel theorem states that  $f = f^{**}$  if and only if  $f$  is convex lower semicontinuous on  $X$ .

The "if" part of Theorem 1 is given by the following theorem.

**Theorem 3.** *Let  $Y$  and  $X$  be Banach spaces and let  $T : Y \rightarrow X$  be a linear continuous operator. Suppose that  $f \circ T$  is Fréchet differentiable at  $y \in Y$  whenever  $f : X \rightarrow \mathbb{R}$  is convex continuous and Gâteaux differentiable at  $T(y) \in X$ . Then  $T$  is a limited operator.*

*Proof.* Let  $(p_n)_n$  be a weak\* null sequence in  $X^*$ . We want to prove that  $\|T^*(p_n)\|_{Y^*} \rightarrow 0$ . Set

$$K = \overline{\text{co}}^{w^*} \{p_n : n \in \mathbb{N}\}.$$

According to Lemma 2,  $K$  is convex weak\* compact and norm separable. Using Proposition 1, there exists a continuous seminorm which is weak\* lower semicontinuous and sequentially weak\* continuous  $h : X^* \rightarrow \mathbb{R}$  such that the restriction  $h|_K$  of  $h$  to  $K$  has a weak\*-strong minimum at 0 and in particular  $\min_K h = h(0) = 0$ . Since the sequence  $(p_n)_n$  weak\* converges to 0, it follows that  $\lim_n h(p_n) = h(0) = \min_K h$ . Thus,  $(p_n)_n$  is a minimizing sequence for  $h|_K$ . Set  $g = h + \delta_K$ , where  $\delta_K$  denotes the indicator function, which is equal to 0 on  $K$  and equal to  $+\infty$  otherwise. Since  $K$  is convex, weak\*-closed and norm bounded, then  $g$  is a convex and

weak\* lower semicontinuous function with a norm bounded domain  $\text{dom}(g) = K$ . Moreover we have,

- (1)  $g(p) > 0 = g(0) = \min_{X^*}(g)$  for all  $p \in X^* \setminus \{0\}$ .
- (2)  $\lim_{n \rightarrow +\infty} g(p_n) = \min_{X^*}(g)$ .

Hence, there exists a convex and Lipschitz continuous function  $f : X \rightarrow \mathbb{R}$  such that  $g = f^*$  (we can take  $f = g|_X$ ). The function  $f$  is Gâteaux differentiable at 0 with Gâteaux derivative  $\nabla f(0) = 0$ , this is due to the fact that  $f^* = g$  has a weak\*-strong minimum at 0 (we can see [Corollary 1. [2]]). Thus, from our hypothesis,  $f \circ T$  is Fréchet differentiable at 0 with Fréchet derivative equal to 0. It follows that  $(f \circ T)^*$  has a norm-strong minimum at 0 (see [Corollary 2. [2]]). Now, we prove that  $(T^*(p_n))_n$  is a minimizing sequence for  $(f \circ T)^*$ , which will implies that  $\|T^*(p_n)\|_{Y^*} \rightarrow 0$ . Indeed, on one hand, we have  $0 = \min_{X^*}(g) = -g^*(0) = -f(0)$ . On the other hand we have

$$\begin{aligned} 0 = -f(0) &\leq \sup_{y \in Y} \{-f \circ T(y)\} &:= (f \circ T)^*(0) \\ &\leq \sup_{x \in X} \{-f(x)\} \\ &= f^*(0) \\ &= g(0) \\ &= 0. \end{aligned}$$

It follows that  $(f \circ T)^*(0) = 0$ . Hence, since  $(f \circ T)^*$  has a minimum at 0, we obtain

$$\begin{aligned} 0 = (f \circ T)^*(0) &\leq (f \circ T)^*(T^*(p_n)) &:= \sup_{y \in Y} \{\langle T^*(p_n), y \rangle - f \circ T(y)\} \\ &= \sup_{y \in Y} \{\langle p_n, T(y) \rangle - f(T(y))\} \\ &\leq \sup_{x \in X} \{\langle p_n, x \rangle - f(x)\} \\ &= f^*(p_n) \\ &= g(p_n). \end{aligned}$$

Since,  $g(p_n) \rightarrow 0$ , it follows that  $(f \circ T)^*(T^*(p_n)) \rightarrow 0 = (f \circ T)^*(0)$ . In other words,  $(T^*(p_n))_n$  is a minimizing sequence for  $(f \circ T)^*$ . Since  $(f \circ T)^*$  has a norm-strong minimum at 0, we obtain that  $\|T^*(p_n)\|_{Y^*} \rightarrow 0$ , which implies that  $T$  is a limited operator.  $\square$

The "only if" part of Theorem 1 is given by the following theorem.

**Theorem 4.** *Let  $Y$  and  $X$  be two Banach spaces and  $T : Y \rightarrow X$  be a limited operator. Let  $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ , be a convex lower semicontinuous function and let  $a \in Y$  such that  $T(a)$  belongs to the interior of  $\text{dom}(f)$ . Then,  $f \circ T$  is Fréchet differentiable at  $a \in Y$  with Fréchet-derivative  $T^*(Q) \in Y^*$ , whenever  $f$  is Gâteaux differentiable at  $T(a) \in X$  with Gâteaux-derivative  $Q \in X^*$ .*

*Proof.* Since  $f$  is convex lower semicontinuous and  $T(a)$  is in the interior of  $\text{dom}(f)$ , there exists  $r_a > 0$  and  $L_a > 0$  such that  $f$  is  $L_a$ -Lipschitz continuous on the closed ball  $B_X(T(a), r_a)$ . It is well known that there exists a convex  $L_a$ -Lipschitz continuous function  $\tilde{f}_a$  on  $X$  such that  $\tilde{f}_a = f$  on  $B_X(T(a), r_a)$  (See for instance Lemma 2.31 [9]). It follows that  $\tilde{f}_a \circ T = f \circ T$  on  $B_Y(a, \frac{r_a}{\|T\|})$ , since  $T(B_Y(a, \frac{r_a}{\|T\|}))$  is a subset of  $B_X(T(a), r_a)$  (we can assume that  $T \neq 0$ ). Replacing  $f$  by  $\frac{1}{L_a} \tilde{f}_a$ , we can assume without loss of generality that  $f$  is convex 1-Lipschitz continuous on  $X$ . It follows that  $\text{dom}(f^*) \subset B_{X^*}$  (the closed unit ball of  $X^*$ ).

*Claim.* Suppose that  $f$  is Gâteaux differentiable at  $T(a) \in X$  with Gâteaux-derivative  $Q \in X^*$ , then the function  $q \mapsto f^*(q) - \langle q, T(a) \rangle$  has a weak\*-strong minimum on  $B_{X^*}$  at  $Q$ .

*Proof of the claim.* See [Corollary 1. [2]].

Now, suppose by contradiction that  $T^*(Q)$  is not the Fréchet derivative of  $f \circ T$  at  $a$ . There exist  $\varepsilon > 0$ ,  $t_n \rightarrow 0^+$  and  $h_n \in Y$ ,  $\|h_n\|_Y = 1$  such that for all  $n \in \mathbb{N}^*$ ,

$$f \circ T(a + t_n h_n) - f \circ T(a) - \langle T^*(Q), t_n h_n \rangle > \varepsilon t_n. \quad (1)$$

Let  $r_n = t_n/n$  for all  $n \in \mathbb{N}^*$  and choose  $p_n \in B_{X^*}$  such that

$$f^*(p_n) - \langle p_n, T(a + t_n h_n) \rangle < \inf_{p \in B_{X^*}} \{f^*(p) - \langle p, T(a + t_n h_n) \rangle\} + r_n. \quad (2)$$

From (2) we get

$$f^*(p_n) - \langle p_n, T(a) \rangle < \inf_{p \in B_{X^*}} \{f^*(p) - \langle p, T(a) \rangle\} + 2t_n \|T\| + r_n.$$

This implies that the sequence  $(p_n)_n$  minimize the function  $q \mapsto f^*(q) - \langle q, T(a) \rangle$  on  $B_{X^*}$ . Using the claim, the function  $q \mapsto f^*(q) - \langle q, T(a) \rangle$  has a weak\*-strong minimum on  $B_{X^*}$  at  $Q$ , it follows that  $(p_n)_n$  weak\* converges to  $Q$  and so (since  $T$  is limited) we have

$$\|T^*(p_n - Q)\|_{Y^*} \rightarrow 0. \quad (3)$$

On the other hand, since  $f(T(a + t_n h_n)) = f^{**}(T(a + t_n h_n)) = -\inf_{p \in B_{X^*}} \{f^*(p) - \langle p, T(a + t_n h_n) \rangle\}$ , using (2) we obtain for all  $y \in Y$

$$\begin{aligned} f \circ T(a + t_n h_n) - \langle p_n, T(a + t_n h_n) \rangle &< -f^*(p_n) + r_n \\ &\leq f \circ T(y) - \langle p_n, T(y) \rangle + r_n. \end{aligned}$$

Replacing  $y$  by  $a$  in the above inequality we obtain

$$f \circ T(a + t_n h_n) - \langle p_n, T(t_n h_n) \rangle \leq f \circ T(a) + r_n. \quad (4)$$

Combining (1) and (4) we get

$$\begin{aligned} \varepsilon &< \langle p_n, T(h_n) \rangle - \langle T^*(Q), h_n \rangle + r_n/t_n \\ &= \langle T^*(p_n), h_n \rangle - \langle T^*(Q), h_n \rangle + \frac{1}{n} \\ &\leq \|T^*(p_n - Q)\|_{Y^*} + \frac{1}{n} \end{aligned}$$

which is a contradiction with (3). Thus  $f \circ T$  is Fréchet differentiable at  $a$  with Fréchet derivative  $T^*(Q)$ .  $\square$

Now, we give the proof of Theorem 2.

**Proof of Theorem 2.** (1)  $\implies$  (2) is the deeper Josefson-Nissenzweig theorem [[7], Chapter XII].

(2)  $\implies$  (1) is well known.

(2)  $\implies$  (3) Let  $(p_n)_n$  be a weak\* null sequence in  $Y^*$  such that  $\inf_n \|p_n\| > 0$  and set  $K = \overline{\text{co}}^{w^*} \{p_n : n \in \mathbb{N}\}$ . By Lemma 2, the set  $K$  is convex norm separable and weak\* compact

metrizable. On the other hand, from Proposition 1, there exists a continuous seminorm  $h$  which is weak\* lower semicontinuous and weak\* sequentially continuous on  $Y^*$  such that the restriction of  $h$  to  $K$  has a weak\*-strong minimum at 0. It remains to show that 0 is not a norm-strong minimum for  $h|_K$ . Indeed, since  $(p_n)_n$  is weak\* null and  $h$  is weak\* sequentially continuous, then  $\lim_n h(p_n) = h(0) = \min_K h$ . So  $(p_n)_n$  is a minimizing sequence for  $h|_K$  which not converges to 0 since  $\inf_n \|p_n\| > 0$ . Hence, 0 is not a norm-strong minimum for  $h|_K$ .

(3)  $\implies$  (2) Since 0 is not a norm-strong minimum for the restriction  $h|_K$ , there exists a sequence  $(p_n)_n$  that minimize  $h$  on  $K$  but  $\|p_n\| \not\rightarrow 0$ . Since  $h|_K$  has a weak\*-strong minimum at 0, it follows that  $(p_n)_n$  weak\* converges to 0. Hence,  $(p_n)_n$  weak\* converges to 0 but  $\|p_n\| \not\rightarrow 0$ . Thus, there exists a JN-sequence in  $Y^*$ .

(2)  $\implies$  (4) This part is given by taking  $X = Y$  and  $T = I$  the identity map. Indeed, there exists a sequence  $(p_n)_n$  which weak\* converges to 0 but  $\inf_n \|I^*(p_n)\| = \inf_n \|p_n\| > 0$ . So  $I$  cannot be a limited operator.

(4)  $\implies$  (5). Indeed, if there exists a Banach space  $X$  and a non-limited operator  $T : Y \rightarrow X$ , by using Theorem 1, there exists a convex continuous function  $f : X \rightarrow \mathbb{R}$  and a point  $y \in Y$  such that  $f$  is Gâteaux differentiable at  $T(y) \in X$  but  $f \circ T$  is not Fréchet differentiable at  $y$ . So  $f \circ T$  is Gâteaux but not Fréchet differentiable at  $y$ . Hence,  $f \circ T$  is a convex continuous PGNF-function on  $Y$ .

(5)  $\implies$  (2) Let  $f$  be a PGNF-function on  $Y$ . We can assume without loss of generality that  $f$  is Gâteaux differentiable at 0 with Gâteaux-derivative equal to 0, but  $f$  is not Fréchet differentiable at 0. It follows from classical duality result (see Corollary 1. and Corollary 2. in [2]) that  $f^*$  has a weak\*-strong minimum but not norm-strong minimum at 0. Since 0 is not a norm-strong minimum for  $f^*$ , there exists a sequence  $(p_n)_n \in X^*$  minimizing  $f^*$  such that  $\|p_n\| \not\rightarrow 0$ . On the other hand, since  $f^*$  has a weak\*-strong minimum at 0, and  $(p_n)_n$  minimize  $f^*$ , we have that  $(p_n)_n$  weak\* converges to 0. Thus,  $(p_n)_n$  weak\* converges to 0 but  $\|p_n\| \not\rightarrow 0$ . Hence, there exists a JN-sequence.  $\square$

**Canonical construction of PGNF-function.** There exist different way to build a PGNF-function in infinite dimensional Banach spaces. We can find examples of such constructions in [5]. We present below a different method for constructing a PGNF-function on a Banach space  $X$  canonically from a JN-sequence. Given a JN-sequence  $(p_n)_n \subset X^*$ , we set  $K = \overline{\text{co}}^{w^*} \{p_n : n \in \mathbb{N}\}$ . Using Lemma 2, there exists a sequence  $(x_n)_n \in S_X$  which separates the points of  $K$ , and as in the proof of Proposition 1, there exist a continuous seminorm  $h$  which is weak\* lower semicontinuous and weak\* sequentially continuous such that  $h|_K$  has a weak\*-strong minimum at 0. The function  $h$  is given explicitly as follows

$$h(x^*) = \left( \sum_{n \geq 0} 2^{-n} (\langle x^*, x_n \rangle)^2 \right)^{\frac{1}{2}}, \quad \forall x^* \in X^*.$$

Since  $(p_n)_n$  weak\* converges to 0, it follows that  $(p_n)$  is a minimizing sequence for  $h|_K$ . Since  $(p_n)_n$  is a JN-sequence, it follows that 0 is not a norm-strong minimum for  $h|_K$ . Define the function  $f$  by

$$f(x) = (h + \delta_K)^*(\hat{x}), \quad \forall x \in X,$$

where  $\delta_K$  denotes the indicator function, which is equal to 0 on  $K$  and equal to  $+\infty$  otherwise and where for each  $x \in X$ , we denote by  $\hat{x} \in X^{**}$  the linear map  $x^* \mapsto \langle x^*, x \rangle$  for all  $x^* \in X^*$ . Then  $f$  is convex Lipschitz continuous, Gâteaux differentiable at 0 (since  $h + \delta_K$  has a weak\*-strong minimum) but not Fréchet differentiable at 0 (because 0 is not a norm-strong minimum for  $h + \delta_K$ ).

## 4 Appendix.

There exists a class of Banach spaces  $(E, \|\cdot\|_E)$  such that the canonical embedding  $i : E \rightarrow E^{**}$  is a limited operator. This class contains in particular the space  $c_0$  and any closed subspace



$F$  of  $c_0$  (This class is also stable by product and quotient. For more information see [6]). In this setting, Theorem 4 gives immediately the following corollary.

**Corollary 1.** *Suppose that the canonical embedding  $i : E \longrightarrow E^{**}$  is a limited operator. Let  $g : E^{**} \longrightarrow \mathbb{R} \cup \{+\infty\}$  be a convex lower semicontinuous function. Suppose that  $x \in E$  belongs to the interior of  $\text{dom}(g)$  and that  $g$  is Gâteaux differentiable at  $x \in E$  (we use the identification  $i(x) = x$ ), then the restriction of  $g$  to  $E$  is Fréchet differentiable at  $x$ . In particular, if  $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$  is convex lower semicontinuous function,  $x \in E$  belongs to the interior of  $\text{dom}((f^*)^*)$  and  $(f^*)^*$  is Gâteaux differentiable at  $x$ , then  $f$  is Fréchet differentiable at  $x$ .*

We obtain the following corollary by combining Proposition 2 and a delicate result due to Zajicek (see [Theorem 2; [12]]), which say that in a separable Banach space, the set of the points where a convex continuous function is not Gâteaux differentiable, can be covered by countably many *d.c* (that is, delta-convex) *hypersurface*. Recall that in a separable Banach space  $Y$ , each set  $A$  which can be covered by countably many *d.c hypersurface* is  $\sigma$ -lower porous, also  $\sigma$ -directionally porous; in particular it is both Aronszajn (equivalent to Gauss) null and  $\Gamma$ -null. For details about this notions of small sets we refer to [13] and references therein. Note that a limited set in a separable Banach space is relatively compact [4].

**Proposition 2.** *Let  $Y$  and  $X$  be Banach spaces and  $T : Y \longrightarrow X$  be a limited operator with a dense range. Let  $f : X \longrightarrow \mathbb{R}$  be a convex continuous function. Then  $f \circ T$  is Gâteaux differentiable at  $a \in Y$  if and only if,  $f \circ T$  is Fréchet differentiable at  $a \in Y$ .*

*Proof.* Suppose that  $f \circ T$  is Gâteaux differentiable at  $a \in Y$ . It follows that  $f$  is Gâteaux differentiable at  $T(a)$  with respect to the direction  $T(Y)$  which is dense in  $X$ . It follows (from a classical fact on locally Lipschitz continuous functions) that  $f$  is Gâteaux differentiable at  $T(a)$  on  $X$ . So by Theorem 4,  $f \circ T$  is Fréchet differentiable at  $a \in Y$ . The converse is always true.  $\square$

**Corollary 2.** *Let  $Y$  be a separable Banach space,  $X$  be a Banach spaces and  $T : Y \longrightarrow X$  be a compact operator with a dense range. Let  $f : X \longrightarrow \mathbb{R}$ , be a convex and continuous function. Then, the set of all points at which  $f \circ T$  is not Fréchet differentiable can be covered by countably many *d.c hypersurface*.*

## References

- [1] K. T. Andrews, *Dunford-Pettis sets in the space of Bochner integrable functions*, Math. Ann. 241, (1979), 35-41.
- [2] E. Asplund and R. T. Rockafellar, *Gradients of convex functions*. Trans. Amer. Math. Soc. 139, (1969), 443-467.
- [3] E. Bishop and R. R. Phelps, *A proof that every Banach space is subreflexive*, Bull. Amer. Math. Soc. 67, (1961), 97-98 .
- [4] J. Bourgain and J. Diestel, *Limited operators and strict cosingularity*, Math. Nachr. 119, (1984), 55-58.
- [5] J. M. Borwein and M. Fabian, *On convex functions having points of Gâteaux differentiability which are not points of Fréchet-differentiability*. Can. J. Math. Vol.45 (6), 1993, 1121-1134.
- [6] H. Carrión, P. Galindo, and M.L Lourenco, *Banach spaces whose bounded sets are bounding in the bidual* Annales Academiae Scientiarum Fennicae Mathematica Volumen 31, (2006), 61-70.

- [7] J. Diestel, *Sequences and series in Banach spaces*, Graduate texts in Mathematics, Springer Verlag, N.Y., Berlin, Tokyo, 1984.
- [8] R. Haydon, *An extreme point criterion for separability of a dual Banach space, and a new proof of a theorem of Corson*, Quart. J. Math. Oxford Ser. 27 (1976), 379-385.
- [9] R. R. Phelps, *Convex Functions, Monotone Operators and Differentiability*. Lecture Notes in Mathematics 1364, Springer-Verlag, Berlin (1993).
- [10] R. R. Phelps, *Lectures on Chaquet's theorem*, Van Nostrand, Princeton, N. J., 1966.
- [11] Th. Schlumprecht, *Limited sets in Banach spaces*, Dissertation, München, 1987.
- [12] L. Zajicek, *On the differentiation of convex functions in finite and infinite dimensional spaces*, Czechoslovak Math. J. 29, (1979), no. 3, 340-348.
- [13] L. Zajicek, *On sigma-porous sets in abstract spaces*, Abstract and Applied Analysis, vol. (2005), issue 5, 509-534,